A criterion for good reduction of Drinfeld modules and Anderson motives in terms of local shtukas

U. Hartl, S. Hüsken*

Abstract

For an Anderson A-motive over a discretely valued field whose residue field has A-characteristic ε, we prove a criterion for good reduction in terms of its associated local shtuka at ε. This yields a criterion for good reduction of Drinfeld modules. Our criterion is the function-field analog of Grothendieck's [SGA 7, Proposition IX.5.13] and de Jong's [dJ98, 2.5] criterion for good reduction of an abelian variety over a discretely valued field with residue characteristic p in terms of its associated p-divisible group.
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Contents

1 Introduction

We fix a finite field $\mathbb{F}$ with r elements and characteristic p. Let $\mathcal{C}$ be a smooth and geometrically irreducible projective curve over $\mathbb{F}$ with function field $Q = \mathbb{F}(\mathcal{C})$. Let $\infty \in \mathcal{C}$ be a closed point and let $A = \Gamma(\mathcal{C} - \{\infty\}, \mathcal{O}_\mathcal{C})$ be the $\mathbb{F}$-algebra of those rational functions on $\mathcal{C}$ which are regular outside ∞. For every $\mathbb{F}$-algebra R we let σ be the endomorphism of $A_R := A \otimes_\mathbb{F} R$ given by $\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{r,R} \colon a \otimes b \mapsto a \otimes b^r$ for $a \in A$ and $b \in R$.

Let o_L be a complete discrete valuation ring containing $\mathbb{F}$, with fraction field L, uniformizing parameter π, maximal ideal $\mathfrak{m}_L = (\pi)$ and residue field $\ell = o_L/\mathfrak{m}_L$. We assume that ℓ is a finite field extension of ℓ^p. This is equivalent to saying that ℓ has a finite p-basis over ℓ^p in the sense of [Bou81, § V.13, Definition 1]. It holds for example if ℓ is perfect, or if ℓ is a finitely generated field. Since every Anderson A-motive over L can be defined over a finitely generated subfield of L our restriction on ℓ is not serious. Let $c^* \colon A \to o_L$ be a homomorphism of $\mathbb{F}$-algebras such that the kernel of the composition $A \to o_L \twoheadrightarrow \ell$ is a *maximal* ideal ε in A. We say that *the residue field ℓ has finite A-characteristic ε*. We do not assume that $c^* \colon A \to o_L$ is injective. So L can have either generic A-characteristic or finite A-characteristic ε. In the rings A_L and A_{o_L} we consider the ideals $\mathfrak{J} := (a \otimes 1 - 1 \otimes c^*(a) \colon a \in A)$.

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By an *Anderson A-motive over L* we mean a pair $\underline{M} = (M, F_M)$ consisting of a locally free A_L-module M of finite rank, and an injective A_L-homomorphism $F_M\colon \sigma^*M \to M$ where $\sigma^*M := M \otimes_{A_L,\sigma} A_L$, such that $\operatorname{coker}(F_M)$ is a finite dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. We say that $\underline{M}$ has *good reduction over o_L* if there exists a locally free A_{o_L}-module $\mathcal{M}$ and an injective A_{o_L}-homomorphism $F_{\mathcal{M}}\colon \sigma^*\mathcal{M} \to \mathcal{M}$ such that $(\mathcal{M}, F_{\mathcal{M}}) \otimes_{A_{o_L}} A_L \cong \underline{M}$ and $\operatorname{coker}(F_{\mathcal{M}})$ is a finite free o_L-module which is annihilated by a power of $\mathfrak{J}$. We call $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ a *good model of $\underline{M}$*. In particular if $\underline{M} = \underline{M}(\varphi)$ is the Anderson A-motive associated with a Drinfeld module φ over L, then $\underline{M}$ has good reduction if and only if φ has good reduction; see Proposition 4.9.

Anderson A-motives are function-field analogs of abelian varieties. For an abelian variety $\mathcal{A}$ over a discretely valued field K with residue field of characteristic p there are criteria for good reduction in terms of local data. For a prime number $l \neq p$ the criterion of Néron-Ogg-Shavarevich [ST68, §1, Theorem 1] states that $\mathcal{A}$ has good reduction if and only if the l-adic Tate module $T_l\mathcal{A}$ of $\mathcal{A}$ is unramified as a $\operatorname{Gal}(K^{\mathrm{alg}}/K)$-representation. At the prime p the criterion of Grothendieck [SGA 7, Proposition IX.5.13] (for $\operatorname{char}(K) = 0$), respectively de Jong [dJ98, 2.5] (for $\operatorname{char}(K) = p$) states that $\mathcal{A}$ has good reduction if and only if the Barsotti-Tate group $\mathcal{A}[p^\infty]$ has good reduction.

These criteria have function-field analogs for Anderson A-motives. The analog of the Néron-Ogg-Shavarevich-criterion was proved by Gardeyn [Gar02, Theorem 1.1]. In this article we simultaneously prove the analog of Grothendieck's and de Jong's criterion. Here the function-field analogs of Barsotti-Tate groups are local shtukas [Har09, §§ 3.1, 3.2] which are defined as follows. Let $A_{o_L,(\varepsilon,\pi)}$ be the (ε, π)-adic completion of A_{o_L}. An (*effective*) *local shtuka at ε over o_L* is a pair $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ consisting of a finite free $A_{o_L,(\varepsilon,\pi)}$-module $\hat{M}$ and an injective $A_{o_L,(\varepsilon,\pi)}$-homomorphism $F_{\hat{M}}\colon \sigma^*\hat{M} \to \hat{M}$ such that $\operatorname{coker}(F_{\hat{M}})$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$. The local shtuka associated with a good model $\underline{\mathcal{M}}$ of an Anderson A-motive is $\underline{\hat{M}}(\underline{\mathcal{M}}) := \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,(\varepsilon,\pi)}$. Our analog of Grothendieck's and de Jong's reduction criterion is now the following

Corollary 6.6. *Let $\underline{M}$ be an Anderson A-motive over L. Then $\underline{M}$ has good reduction over o_L if and only if there is an effective local shtuka $\underline{\hat{M}}$ at ε over o_L and an isomorphism $\underline{M} \otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}}[1/\pi]$.*

This applies in particular if $\underline{M}$ is the Anderson A-motive associated with a Drinfeld module φ over L to give a criterion for good reduction of φ in terms of its associated local shtuka.

2 The base rings

Let o_L be an equi-characteristic complete discrete valuation ring containing the finite field $\mathbb{F}$, with quotient field $L = \operatorname{Frac}(o_L)$ and residue field $\ell = o_L/\mathfrak{m}_L$, where $\mathfrak{m}_L \subseteq o_L$ is the maximal ideal of o_L. We assume that ℓ is a *finite* field extension of $\ell^p := (b^p\colon b \in \ell)$. We fix a uniformizer $\pi = \pi_L$ of o_L and sometimes identify o_L with $\ell[\![\pi]\!]$. Let $v = v_\pi = \operatorname{ord}_\pi(\cdot)$ be the discrete valuation on L normalized by $v(\pi) = 1$.

We assume that there is an o_L-valued point $c \in \mathcal{C}(o_L)$ such that the corresponding $\mathbb{F}$-morphism $c\colon \operatorname{Spec}(o_L) \to \mathcal{C}$ factors via $\mathcal{C} - \{\infty\} \subseteq \mathcal{C}$. Such a datum corresponds to a homomorphism of $\mathbb{F}$-algebras $c^*\colon A \to o_L$ which we call the *characteristic map*. We further assume that the closed point $V(\pi) \subseteq \operatorname{Spec}(o_L)$ is mapped to a closed point ε of $\operatorname{Spec}(A) \subseteq \mathcal{C}$. The latter is the kernel of the composition $A \to o_L \twoheadrightarrow \ell$. So, in accordance with Drinfeld's terminology [Dri76], we call ε the *residue characteristic* or *residual characteristic place of Q*. By continuity, the characteristic map $c^*\colon A \to o_L$ factors through to a morphism of complete discrete valuation rings $A_\varepsilon \to o_L$ where A_ε is the completion of A at the characteristic place ε. Note that $A_\varepsilon \to o_L$ is injective if c^* is injective, and factors through A/ε if c^* is not injective.

Remark 2.1. Since A is a Dedekind domain there is a power ε^m which is a principal ideal in A. We fix a generator t of ε^m and frequently use the finite flat monomorphism of $\mathbb{F}$-algebras $\iota\colon \mathbb{F}[z] \to A, z \mapsto t$.

For any $\mathbb{F}$-algebra R we abbreviate $A_R := A \otimes_{\mathbb{F}} R$. In particular, $A_{o_L} \subseteq A_L$ is a noetherian integral domain, and by virtue of the equality $A_\ell \cong A_{o_L}/\pi A_{o_L}$ it follows that $\pi \in o_L$ is a prime element of A_{o_L}.

Definition 2.2. Let $A_{o_L,\pi}$ (resp., $A_{o_L,(\varepsilon,\pi)}$) be the completion of the o_L-algebra A_{o_L} for the π-adic topology (resp., the (ε,π)-adic topology).

By Krull's Theorem ([Bou67], III.3.2), the ring A_{o_L} is separated for both the π-adic and the (ε,π)-adic topology. The topological o_L-algebra $A_{o_L,\pi}$ is admissible in the sense of Raynaud, i.e. it is of topologically finite presentation and has no π-torsion. In particular, the L-algebra $A_{o_L,\pi}[1/\pi]$ is affinoid in the sense of rigid analytic geometry; see [Bos08, BL93a, BGR84].

For example if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $A = \mathbb{F}[z]$ then we have $A_{o_L} = o_L[z]$ and correspondingly $A_L = L[z]$. Let us specify that $\varepsilon = z\mathbb{F}[z]$. Our choice of a uniformizer π gives rise to an identification $o_L = \ell[\![\pi]\!]$. Consequently $o_L[\![z]\!] = \ell[\![\pi]\!][\![z]\!] = \ell[\![\pi, z]\!] = A_{o_L,(\varepsilon,\pi)}$. On the other hand, the π-adic completion of $o_L[z]$ equals $o_L\langle z\rangle := \{\sum_{i=0}^\infty b_i z^i \colon v(b_i) \to \infty (i \to \infty)\}$, and since $L\langle z\rangle = o_L\langle z\rangle \otimes_{o_L} L$, we may view $A_{o_L,\pi}[1/\pi]$ as a replacement, for general $\mathcal{C}$, of the Tate algebra $L\langle z\rangle$ of strictly convergent power series in one indeterminate z over L, which serves as coordinate ring for the one-dimensional affinoid unit ball in rigid analytic geometry.

There is a natural embedding $A_L \to A_{o_L,\pi}[1/\pi]$ which, for general $\mathcal{C}$, replaces the completion homomorphism $L[z] \to L\langle z\rangle$, and which itself can be regarded as a completion map with respect to the L-algebra norm-topology on the *reduced* affinoid L-algebra $A_{o_L,\pi}[1/\pi]$ and its restriction on A_L; see [Bos08, §1.4, Proposition 19]. Note that the canonical homomorphism $A_{o_L} \to A_{o_L,(\varepsilon,\pi)}$ factors uniquely via $A_{o_L,\pi}$, where the induced map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ identifies $A_{o_L,(\varepsilon,\pi)}$ with the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion of $A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is a regular integral domain, it is $(\varepsilon,\pi)A_{o_L,\pi}$-adically separated by Krull's theorem and $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is injective and flat.

Recall that there is a finite flat monomorphism of $\mathbb{F}$-algebras $\iota\colon \mathbb{F}[z] \to A$ which identifies the indeterminate z with the generator $t \in A$ of ε^m chosen in Remark 2.1. The o_L-algebra homomorphism $\iota \otimes \mathrm{id}\colon o_L[z] \to A_{o_L}$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu t^\nu \otimes a_\nu$, is finite flat, so that we obtain finite flat maps

$$o_L\langle z\rangle \to A_{o_L,\pi}, \quad L\langle z\rangle \to A_{o_L,\pi}[1/\pi], \quad o_L[\![z]\!] \to A_{o_L,(t,\pi)}, \quad \ell[z] \to A_\ell. \tag{2.1}$$

Here the (t,π)-adic completion $A_{o_L,(t,\pi)}$ of A_{o_L} equals $A_{o_L,(\varepsilon,\pi)}$ since $(\varepsilon,\pi)^m \subseteq (\varepsilon^m,\pi) = (t,\pi)$ in A_{o_L}.

Lemma 2.3. *If $A_{o_L,\varepsilon}$ denotes the ε-adic completion of A_{o_L}, the canonical map $A_{o_L,\varepsilon} \to A_{o_L,(\varepsilon,\pi)}$ is an isomorphism.* □

3 Frobenius modules

The r-Frobenius $\mathrm{Frob}_r\colon o_L \to o_L$, $x \mapsto x^r$, gives rise to an endomorphism

$$\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_r\colon A_{o_L} \to A_{o_L}, \quad a \otimes x \mapsto a \otimes x^r,$$

which extends to give a map $\mathrm{id}_A \otimes \mathrm{Frob}_{r,L}\colon A_L \to A_L$ again denoted by σ. On the other hand, reducing mod π gives $\bar\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_{r,\ell}\colon A_\ell \to A_\ell$. The latter is a finite flat endomorphism of the Dedekind domain A_ℓ, because ℓ is finite over ℓ^p. The map $\sigma\colon A_{o_L} \to A_{o_L}$ is π-adically and (ε,π)-adically continuous and therefore extends to give endomorphisms $A_{o_L,\pi} \to A_{o_L,\pi}$ and $A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$, again denoted by σ.

Lemma 3.1. *In the commutative diagram*

$$\begin{tikzcd} A_{o_L} \arrow[r] \arrow[d, "\sigma"'] & A_{o_L,\pi} \arrow[r] \arrow[d, "\sigma"'] & A_{o_L,(\varepsilon,\pi)} \arrow[d, "\sigma"'] \\ A_{o_L} \arrow[r] & A_{o_L,\pi} \arrow[r] & A_{o_L,(\varepsilon,\pi)} \end{tikzcd}$$

both squares are cocartesian, and the vertical arrows are finite flat.

We let the proof be preceded by the following

Remark. Via the identification $o_L = \ell[\![\pi]\!]$, the r-Frobenius $\mathrm{Frob}_{r,o_L}\colon o_L \to o_L$ is mirrored by the map $\ell[\![\pi]\!] \to \ell[\![\pi]\!]$, $\sum_{\nu=0}^{\infty} a_\nu \pi^\nu \mapsto \sum_{\nu=0}^{\infty} a_\nu^r \pi^{r\nu}$. Choosing an ℓ^r-basis of ℓ and lifting it to a subset W of o_L, this implies $(\mathrm{Frob}_{r,o_L})_* o_L = \bigoplus_{i=0}^{r-1} \bigoplus_{w\in W} o_L\, w\pi^i$, so that $\mathrm{Frob}_{r,o_L}\colon o_L \to o_L$ is finite flat.

Proof of Lemma 3.1. By base change the remark implies that $\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_{r,o_L}\colon A_{o_L} \to A_{o_L}$ is finite flat, and that $A_{o_L} \otimes_{\sigma, A_{o_L}} A_{o_L,\pi}$ is a finite flat $A_{o_L,\pi}$-module and hence equals the π-adic completion of the A_{o_L}-module $\sigma_* A_{o_L}$. If we let $\mathfrak{a} = \sigma(\pi A_{o_L})A_{o_L} = \pi^r A_{o_L}$ and $\mathfrak{b} = \pi A_{o_L}$, we get $\mathfrak{b}^r = \mathfrak{a} \subseteq \mathfrak{b}$. Consequently, by [Eis95, Lemma 7.14], the inverse systems $(A_{o_L}/\mathfrak{a}^n)_n$ and $(A_{o_L}/\mathfrak{b}^n)_n$ give the same limit, which shows that the square on the left is cocartesian, and that $\sigma\colon A_{o_L,\pi} \to A_{o_L,\pi}$ is finite flat. Similarly, we have $\sigma(\varepsilon,\pi)A_{o_L} = (\varepsilon, \pi^r) \subseteq (\varepsilon,\pi)$ as well as $(\varepsilon,\pi)^r \subseteq (\varepsilon,\pi^r)$, which proves that the displayed diagram qualifies $A_{o_L,(\varepsilon,\pi)}$ as tensor product $A_{o_L,(\varepsilon,\pi)} \otimes_{A_{o_L},\sigma} A_{o_L}$, and that $\sigma\colon A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$ is finite flat. □

Finally, note that the embedding of o_L-algebras $\iota \otimes \mathrm{id}\colon o_L[z] \to A_{o_L}$ commutes with $\sigma\colon A_{o_L} \to A_{o_L}$ and the r-Frobenius lift of $o_L[z]$, given by $o_L[z] \to o_L[z]$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu a_\nu^r z^\nu$. Consequently, also the embeddings from (2.1) are Frobenius-equivariant.

Let B be an o_L-algebra together with a ring endomorphism $\sigma\colon B \to B$ such that σ and $\mathrm{Frob}_{r,o_L}\colon o_L \to o_L$ are compatible with the structure map $o_L \to B$. For example, B could be any of the base rings considered in the previous sections.

Definition 3.2. We define the category $\mathrm{FMod}(B)$ of *Frobenius B-modules* (or simply *F-modules* over B) as follows:

— An object of $\mathrm{FMod}(B)$ is a pair $\underline{M} = (M, F)$ consisting of an B-module M which is locally free of finite rank, together with an *injective* B-linear map $F = F_M\colon \sigma^* M \to M$, where $\sigma^* M := M \otimes_{B,\sigma} B$.

— A *morphism* of Frobenius B-modules $(M, F_M) \to (N, F_N)$ is an B-linear map $\varphi\colon M \to N$ between the underlying B-modules such that φ is *F-equivariant*, i.e. such that $\varphi \circ F_M = F_N \circ \sigma^*\varphi$. It is called an *isomorphism* if φ is an isomorphism of the underlying B-modules.

Let B' be a flat B-algebra together with a ring endomorphism $\sigma\colon B' \to B'$ extending the Frobenius lift of B, as explained before. Then the exact functor $\cdot \otimes_B B'$ from B-modules to B'-modules yields a functor $\mathrm{FMod}(B) \to \mathrm{FMod}(B')$. If the structure map $B \to B'$ is, in addition, injective then the induced functor on $\mathrm{FMod}(B)$ is faithful since, given a map $f\colon M \to N$ of finite projective B-modules, restricting its image $f \otimes \mathrm{id}\colon M \otimes_B B' \to N \otimes_B B'$ to M gives back f. In particular, we obtain a natural commutative diagram of categories and faithful functors

$$\begin{tikzcd} \mathrm{FMod}(A_{o_L}) \arrow[r] \arrow[d] & \mathrm{FMod}(A_{o_L,\pi}) \arrow[r] \arrow[d] & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}) \arrow[d] \\ \mathrm{FMod}(A_L) \arrow[r] & \mathrm{FMod}(A_{o_L,\pi}[1/\pi]) \arrow[r] & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \end{tikzcd}$$

Slightly abusing notation, we agree to write $\underline{M} \otimes_B B'$ for $(M \otimes_B B', F_M \otimes \mathrm{id}_{B'})$, whenever $\underline{M} = (M, F_M)$.

4 Anderson motives

Let $\mathfrak{J} \subseteq A_{o_L}$ be the ideal generated by $a \otimes 1 - 1 \otimes c^*(a)$ for all $a \in A$. For example, if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $A = \mathbb{F}[z]$, then $\mathfrak{J} = (z - \zeta) \subseteq o_L[z]$ where $\zeta = c^*(z)$. Note that $\zeta = 0$ if c^* is not injective. We consider the following variant of Anderson's [And86] t-motives.

Definition 4.1. An *Anderson A-motive over L* is an object $\underline{M} = (M, F_M) \in \mathrm{FMod}(A_L)$ such that $\mathrm{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. A *morphism* of Anderson A-motives is defined as a morphism inside $\mathrm{FMod}(A_L)$.

Since $\mathrm{Spec}(A_L)$ is of finite type over L, one can consider its rigid analytification $\mathrm{Spec}(A_L)^{\mathrm{an}}$; see [Bos08], [BGR84], [FP04]. In accordance with [BH07], we denote this rigid analytic L-space by $\mathfrak{A}(\infty)$. On the other hand, the formal completion of the o_L-scheme $X = \mathrm{Spec}(A_{o_L})$ along its special fiber $V(\pi)$ leads to the formal o_L-scheme $\mathfrak{X} = \mathrm{Spf}(A_{o_L,\pi})$; see [EGA, $\mathrm{I}_{\mathrm{new}}$, I.10.8.3]. Its associated rigid analytic space $\mathfrak{X}_{\mathrm{rig}}$ ([Bos08], [FP04]) is given by the affinoid L-space $\mathfrak{A}(1) := \mathrm{Sp}(A_{o_L,\pi}[1/\pi])$. This space can be regarded as the unit disc of the rigid analytic space $\mathfrak{A}(\infty)$ as it corresponds to "radius of convergence 1", hence the notation.

We study the following instance of rigid analytic τ-sheaves over $A_{o_L,\pi}[1/\pi]$, in the sense of [BH07].

Definition 4.2. An *analytic Anderson $A(1)$-motive* over L is an object $\underline{M} = (M, F_M) \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ such that $\mathrm{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. A *morphism* of analytic Anderson $A(1)$-motives is defined as a morphism in the category $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

Here the prefix "$A(1)$-" indicates that we are considering an analytic variant of Anderson A-motives over the rigid analytic "unit disc" $\mathfrak{A}(1)$ in $\mathrm{Spec}(A_L)$.

Proposition 4.3. *The natural functor* $\mathrm{FMod}(A_L) \to \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, $\underline{M} \mapsto \underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ *restricts to a functor* (*Anderson A-motives over L*) $\to$ (*analytic Anderson $A(1)$-motives over L*). □

Definition 4.4. (a) Let $\underline{M}_L \in \mathrm{FMod}(A_L)$ be an F-module over A_L. A *model* of $\underline{M}_L$ is an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ such that $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_L \cong \underline{M}_L$ inside $\mathrm{FMod}(A_L)$.

(b) Let $\underline{M}_L \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ be an F-module over $A_{o_L,\pi}[1/\pi]$. A *(formal) model* of $\underline{M}_L$ is an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ such that $\underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi] \cong \underline{M}_L$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

For every $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$, resp. $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ we can consider the reduction $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_\ell$, resp. $\underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_\ell$. Note, however, that this does *not* induce a functor from $\mathrm{FMod}(A_{o_L})$, resp. $\mathrm{FMod}(A_{o_L,\pi})$ to $\mathrm{FMod}(A_\ell)$, since the induced F-map need not be injective. This circumstance lies at the origin of our study of good models:

Definition 4.5. Let $\underline{\mathcal{M}}$ be a model of an F-module $\underline{M}_L$ over A_L, resp. over $A_{o_L,\pi}[1/\pi]$. Then $\underline{\mathcal{M}}$ is called a *good* model if $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ is an F-module over A_ℓ, i.e. if the induced A_ℓ-linear map

$$\bar{\sigma}^*(\mathcal{M}/\pi\mathcal{M}) = (\mathcal{M}/\pi\mathcal{M}) \otimes_{A_\ell,\bar{\sigma}} A_\ell \to \mathcal{M}/\pi\mathcal{M}$$

is injective.

If $\underline{M}_L$ is an (analytic) Anderson motive there is a stronger notion of good reduction as follows.

Definition 4.6. Let $\underline{\mathcal{M}}$ be a model of an Anderson A-motive $\underline{M}_L$, resp. of an analytic Anderson $A(1)$-motive $\underline{M}_L$. Then $\underline{\mathcal{M}}$ is called a *good model in the strong sense* if $\mathrm{coker}(F_{\mathcal{M}})$ is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$, for some $d \geq 0$. In this case we also say that $\underline{\mathcal{M}}$ has *good reduction over o_L*.

Remark 4.7. If $\underline{\mathcal{M}}$ is a good model in the strong sense of an Anderson A-motive $\underline{M}$ then $\underline{\mathcal{M}}$ is also a good model in the sense of Definition 4.5. Indeed, since $\sigma^*\mathcal{M}$ is locally free over A_{o_L} the natural map $\sigma^*\mathcal{M} \to \sigma^*M$ is injective and hence $F_{\mathcal{M}}$ is injective because F_M is. Tensoring the short exact sequence $0 \to \sigma^*\mathcal{M} \xrightarrow{F_{\mathcal{M}}} \mathcal{M} \to \operatorname{coker}(F_{\mathcal{M}}) \to 0$ with ℓ over o_L and using that $\operatorname{coker}(F_{\mathcal{M}})$ is supposed to be free over o_L shows that the induced A_ℓ-linear map $\bar{\sigma}^*(\mathcal{M}/\pi\mathcal{M}) \to \mathcal{M}/\pi\mathcal{M}$ remains injective.

For Anderson A-motives associated with Drinfeld modules the converse also holds; see Proposition 4.9. In general the converse is false.

Proposition 4.8. *If $\underline{M}_L$ is an Anderson A-Motive over L having a good model $\underline{\mathcal{M}}$ then its analytification $\underline{M}_L \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ is an analytic Anderson $A(1)$-motive having the good model $\widehat{\underline{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ and the reduction $\widehat{\underline{\mathcal{M}}}/\pi\widehat{\underline{\mathcal{M}}}$ of $\widehat{\underline{\mathcal{M}}}$ is canonically isomorphic to the reduction $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ of $\underline{\mathcal{M}}$.* □

Proposition 4.9. *Let $\varphi\colon A \to L[\tau]$ be a Drinfeld A-module over L; see [Dri76] or [Mat96]. Let $\underline{M} = \underline{M}(\varphi)$ be the associated Anderson A-motive; see [And86, §4.1] or [Gar02, §8.1]. Then the following are equivalent:*

(i) φ has good reduction over o_L,

(ii) $\underline{M}$ has good reduction over o_L in the strong sense of Definition 4.6,

(iii) $\underline{M}$ has a good model in the weak sense of Definition 4.5.

Proof. The equivalence of (i) and (iii) was proved by Gardeyn [Gar02, Theorem 8.1]. Since (ii) implies (iii) by Remark 4.7 it remains to prove that (i) implies (ii). Denote the rank of φ by ρ and set $n(a) := \rho \operatorname{ord}_\infty(a)$. To say that φ has good reduction means that φ can be written in the form $\varphi\colon A \to L[\tau]$, $a \mapsto \varphi_a$ with

$$\varphi_a = \sum_{i=0}^{n(a)} \delta_{a,i}\tau^i$$

where $\delta_{a,0} = c^*(a)$, $\delta_{a,i} \in o_L$ and $\delta_{a,n\operatorname{ord}_\infty(a)} \in o_L^\times$. We equip $\mathcal{M} := o_L[\tau]$ with the action of A_{o_L} on $m \in \mathcal{M}$ by $a \cdot m := m \cdot \varphi_a$ and $b \cdot m := bm$ for $a \in A$ and $b \in o_L$. Together with the σ-semilinear Frobenius $F_{\mathcal{M}}^{\mathrm{semi}}\colon \mathcal{M} \to \mathcal{M}$, $m \mapsto \tau \cdot m$ and its linearization $F_{\mathcal{M}}\colon \sigma^*\mathcal{M} \to \mathcal{M}$ which satisfies $F_{\mathcal{M}}^{\mathrm{semi}}(m) = F_{\mathcal{M}}(m \otimes 1)$ we obtain a model $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ of the Anderson A-motive $\underline{M}(\varphi)$. To prove that $\underline{\mathcal{M}}$ is a good model in the sense of Definition 4.6, we use the morphism $\mathbb{F}[z] \to A$, $z \mapsto t$ from Remark 2.1. Under the restriction of scalars to $\mathbb{F}[z]$ the $o_L[z]$-module $\mathcal{M}$ is isomorphic to $\bigoplus_{i=0}^{n(t)} o_L\,\tau^i$ with $F_{\mathcal{M}}$ given by the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & (z-\zeta)\delta_{n(t)}^{-1} \\ 1 & \ddots & \vdots & -\delta_1\delta_{n(t)}^{-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \\ 0 & \cdots\ 0 & 1 & -\delta_{r-1}\delta_{n(t)}^{-1} \end{pmatrix}.$$

This shows that $\operatorname{coker}(F_{\mathcal{M}})$ is isomorphic to $o_L[z]/(z-\zeta) = o_L$ and annihilated by $z - \zeta \in \mathfrak{J}$. Therefore $\underline{\mathcal{M}}$ is a good model of $\underline{M}$ in the strong sense of Definition 4.6. □

5 Local shtukas and analytic Anderson motives

Anderson A-motives can be viewed as function-field analogs of abelian varieties. Barsotti-Tate groups, which can be associated with abelian varieties over $\mathbb{Z}_p$-schemes, have effective local shtukas as function-field analogs.

Definition 5.1. An *(effective) local shtuka at ε over o_L* is an object $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}}) \in \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)})$ such that $\mathrm{coker}(F_{\hat{M}})$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$.

Remark 5.2. If the residue field $\mathbb{F}_\varepsilon = A/\varepsilon$ of ε is larger than $\mathbb{F}$, i.e. $d_\varepsilon := [\mathbb{F}_\varepsilon : \mathbb{F}] > 1$, the ring $A_{o_L,(\varepsilon,\pi)}$ is not an integral domain but a product $A_{o_L,(\varepsilon,\pi)} = \prod_{i\in\mathbb{Z}/d_\varepsilon\mathbb{Z}} A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_i$ of integral domains. To describe this product decomposition, note that $A_{o_L,(\varepsilon,\pi)} = \varprojlim_n A_{o_L}/\varepsilon^n = \varprojlim_n (A/\varepsilon^n) \otimes_{\mathbb{F}} o_L = \widehat{A}_\varepsilon \widehat{\otimes}_{\mathbb{F}} o_L$. By Cohen's structure theorem $\widehat{A}_\varepsilon \cong \mathbb{F}_\varepsilon[\![z_\varepsilon]\!]$ for a uniformizer $z_\varepsilon \in A$ of A at ε. Then $\mathfrak{a}_i = (\alpha \otimes 1 - 1 \otimes c^*(\alpha)^{r^i} : \alpha \in \mathbb{F}_\varepsilon \subseteq \widehat{A}_\varepsilon)$, where we use that $c^*\colon A \to o_L$ factors through $c^*\colon \widehat{A}_\varepsilon \to o_L$. The factors $A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_i$ are isomorphic to $o_L[\![z_\varepsilon]\!]$ and hence are integral domains. They are cyclically permuted by σ because $\sigma(\mathfrak{a}_i) = \mathfrak{a}_{i+1}$. By [BH11, Proposition 8.8] the functor $(\hat{M}, F_{\hat{M}}) \mapsto (\hat{M}/\mathfrak{a}_0\hat{M}, (F_{\hat{M}})^{d_\varepsilon})$ is an equivalence between the category of effective local shtukas at ε over o_L as in Definition 5.1 and the category of pairs $(\hat{M}_0, \widetilde{F}_{\hat{M}})$ where $\hat{M}_0$ is a free module of finite rank over $A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_0$ and $\widetilde{F}_{\hat{M}}\colon (\sigma^{d_\varepsilon})^*\hat{M}_0 \to \hat{M}_0$ is injective with $\mathrm{coker}(\widetilde{F}_{\hat{M}})$ being a finite free o_L-module. In [Har09, Har11] these pairs $(\hat{M}_0, \widetilde{F}_{\hat{M}})$ are called *(effective) local shtukas*.

The following criterion for good reduction of analytic Anderson $A(1)$-motives can be regarded as a *good-reduction Local-Global Principle at the characteristic place.*

Theorem 5.3. *Let $\underline{M}_L = (M_L, F_{M_L})$ be an analytic Anderson $A(1)$-motive over L such that $\mathrm{coker}(F_{M_L})$ is annihilated by $\mathfrak{J}^d$ say. Then the following assertions are equivalent:*

(i) $\underline{M}_L$ admits a good model in the strong sense of Definition 4.6.

(ii) There is an effective local shtuka $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ at ε over o_L such that $\mathrm{coker}(F_{\hat{M}})$ is annihilated by $\mathfrak{J}^d$, and an isomorphism $\underline{M}_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$ in $\mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])$.

Proof. In order to show that (ii) implies (i), let $f\colon M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \to \hat{M} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi] =: \hat{M}[1/\pi]$ be an isomorphism of $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-modules as in (ii). We have canonical F-equivariant $A_{o_L,\pi}$-linear maps

$$i\colon M_L \to M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi], \qquad j\colon \hat{M} \to \hat{M}[1/\pi]$$

where i (resp., j) is injective since M_L (resp., $\hat{M}$) is flat. Consider the $A_{o_L,\pi}$-module $\mathcal{M} = \mathrm{im}(i) \cap f^{-1}(\mathrm{im}(j))$. We will show that $\mathcal{M}$ is a good model of $\underline{M}_L$. The inclusion $\mathcal{M} \hookrightarrow M_L$ gives rise to an $A_{o_L,\pi}[1/\pi]$-linear embedding $\mathcal{M}[1/\pi] \hookrightarrow M_L[1/\pi] \cong M_L$, which is in fact an isomorphism, because if $m \in M_L$ there is an $s \geq 0$ such that $\pi^s f(m \otimes 1) \in \mathrm{im}(j)$, i.e. $\pi^s m \otimes 1 \in \mathcal{M}$.

1. First note that $\sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$ because the functor σ^* is exact by Lemma 3.1. By the F-equivariance of f we obtain a Frobenius $F_{\mathcal{M}}\colon \sigma^*\mathcal{M} \to \mathcal{M}$. It is injective because F_{M_L} is. We set $\underline{\mathcal{M}} := (\mathcal{M}, F_{\mathcal{M}})$.

2. Next we claim that $\mathfrak{J}^d \mathrm{coker}(F_{\mathcal{M}}) = 0$, where $\mathfrak{J} = (a \otimes 1 - 1 \otimes c^*(a), a \in A) \subseteq A_{o_L}$. Let $x = \sum_\nu \alpha_\nu m_\nu \in \mathfrak{J}^d\mathcal{M}$ where $\alpha_\nu \in \mathfrak{J}^d$ and $m_\nu \in \mathcal{M}$. Since $\mathrm{coker}(F_{M_L})$ is annihilated by $\mathfrak{J}^d$, there is a (unique) $y \in \sigma^* M_L$ such that $x = \sum_\nu \alpha_\nu m_\nu = F_{M_L}(y)$. We have to show that $y \in \sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$. So it remains to see that $(\sigma^* f)(y) \in \mathrm{im}(\sigma^* j)$. Indeed, inside $\hat{M}[1/\pi]$ we have $f(x) = f(F_{M_L}(y)) = F_{\hat{M}}((\sigma^* f)(y))$. On the other hand, the linearity of f and j gives that $f(x) = \sum_\nu \alpha_\nu f(m_\nu \otimes 1) = j(y')$ for some $y' \in \mathfrak{J}^d\hat{M} \subseteq \mathrm{im}(F_{\hat{M}})$, say $y' = F_{\hat{M}}(y'')$ for a $y'' \in \sigma^*\hat{M}$. Thus $f(x) = F_{\hat{M}}((\sigma^* j)(y''))$. So finally, since $F_{\hat{M}}\colon \sigma^*\hat{M}[1/\pi] \to \hat{M}[1/\pi]$ is injective, we obtain that $(\sigma^* f)(y) = (\sigma^* j)(y'')$, as desired.

3. In order to show that $\mathcal{M}$ is a finitely generated $A_{o_L,\pi}$-module we use the embedding $\iota\colon \mathbb{F}[z] \to A$ from Remark 2.1 and the induced maps $L\langle z\rangle \to A_{o_L,\pi}[1/\pi]$ and $o_L[\![z]\!] \to A_{o_L,(\varepsilon,\pi)}$ from (2.1). Let $(e_1, ..., e_m)$ be a basis of M_L over the principal ideal domain $L\langle z\rangle$. Furthermore, let $(d_1, ..., d_n)$ be a basis for $\hat{M}$ over the local ring $o_L[\![z]\!]$. Note that the basis $(e_1, ..., e_m)$ gives rise to an isomorphism $M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \cong o_L[\![z]\!][1/\pi]^{\oplus m}$.

For every $\nu = 1, ..., n$ we consider $f^{-1}(d_\nu)$ and regard it as an element of the right-hand side of this isomorphism. We choose $N \geq 0$ big enough, such that $f^{-1}(\pi^N d_\nu) \in o_L[\![z]\!]^{\oplus m}$ for all ν, say

$$f^{-1}(\pi^N d_\nu) \;=\; (\rho_{\nu,1}, ..., \rho_{\nu,m})$$

where $\rho_{\nu,\mu} \in o_L[\![z]\!]$. Now let $x \in \mathcal{M}$. Via f we obtain $f(x) = \sum_\nu \lambda_\nu d_\nu$ in $\hat{M}$, with suitable $\lambda_\nu \in o_L[\![z]\!]$. Consequently $f(\pi^N x) = \sum_\nu \lambda_\nu(\pi^N d_\nu)$, so that the image of $\pi^N x$ in $o_L[\![z]\!]^{\oplus m}$ satisfies $\pi^N x = \sum_\mu (\sum_\nu \lambda_\nu \rho_{\nu,\mu}) e_\mu$. The appearing scalars $\alpha_\mu = \sum_\nu \lambda_\nu \rho_{\nu,\mu}$ have, in fact, to be elements of $L\langle z\rangle \cap o_L[\![z]\!] = o_L\langle z\rangle$. Inside M_L we may write $x = \pi^{-N}\pi^N x = \sum_\mu \alpha_\mu \pi^{-N} e_\mu$, so that we may conclude

$$\mathcal{M} \subseteq \sum_\mu o_L\langle z\rangle \pi^{-N} e_\mu.$$

Being a submodule of a finitely generated module over a noetherian ring, $\mathcal{M}$ has to be a finitely generated $o_L\langle z\rangle$-module and hence a finitely generated $A_{o_L,\pi}$-module.

4. We claim that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free and hence free over $\ell[z]$, because it is finitely generated. Let $x \in \mathcal{M}$, and let $\lambda \in o_L\langle z\rangle$ be such that $\lambda \notin \pi o_L\langle z\rangle$ and $\lambda x \in \pi\mathcal{M}$, say $\lambda x = \pi y$ for some $y \in \mathcal{M}$. In order to prove that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free we must show that $x \in \pi\mathcal{M}$. First suppose that $\lambda \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$. We consider $\pi^{-1}x \in M_L$. In fact, this element lies in $\mathcal{M}$, since we have $f(\pi^{-1}x) = \lambda^{-1} f(y) \in \hat{M}$. Consequently $x = \pi(\pi^{-1}x) \in \pi\mathcal{M}$.

Let us next assume that $\lambda = z^n$ and show that $z^n x \in \pi\mathcal{M}$ implies $x \in \pi\mathcal{M}$ for any $n \geq 0$. By induction, it suffices to consider the case $n = 1$. So suppose $zx \in \pi\mathcal{M}$, say $zx = \pi y$. Let $f(x) = \sum_\nu \beta_\nu d_\nu$, where $(d_1, ..., d_n)$ is the finite $o_L[\![z]\!]$-basis of $\hat{M}$ fixed before. The relation $zx = \pi y$ implies that $\pi \mid z\beta_\nu$ for every index ν, so that $\pi \mid \beta_\nu$ for every ν. Therefore $\pi^{-1}x \in M_L$ necessarily maps via f to an element of $\hat{M}$, i.e. $x \in \pi\mathcal{M}$.

Finally we treat the case for general $\lambda = \sum_s \lambda_s z^s$ and suppose that $\lambda \notin o_L[\![z]\!]^\times$, that is $\pi \mid \lambda_0$. This means we find $\lambda' \in o_L[z]$ and $\lambda'' \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$ such that $\lambda = \pi\lambda' + z^N\lambda''$ for some $N \geq 1$. We have $\pi y = \lambda x = \pi\lambda' x + z^N\lambda'' x$. In particular $z^N\lambda'' x = \pi(y - \lambda' x) \in \pi\mathcal{M}$ and by the above $\lambda'' x \in \pi\mathcal{M}$ and $x \in \pi\mathcal{M}$.

Thus we have proved that $\mathcal{M}/\pi\mathcal{M}$ is free over $\ell[z]$. It follows that $\mathcal{M}/\pi\mathcal{M}$ is locally free of finite rank over A_ℓ.

5. We claim that $\mathcal{M}$ is locally free of finite rank over $A_{o_L,\pi}$. Since it is finitely generated it only remains to show that $\mathcal{M}$ is flat over $A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is π-adically complete and separated, $\pi A_{o_L,\pi}$ is contained in the Jacobson radical $\mathfrak{j}(A_{o_L,\pi})$ by [Mat86, Theorem 8.2], and the $A_{o_L,\pi}$-module $\mathcal{M}$ is finitely generated, so that $\mathcal{M}$ is π-adically *ideally Hausdorff* in the sense of [Bou67, III.5.1]. In the preceding step we have shown that $\mathcal{M}/\pi\mathcal{M}$ is flat over $A_\ell \cong A_{o_L,\pi}/\pi A_{o_L,\pi}$, and we know that $\mathcal{M}$ has no π-torsion, so that the canonical map $\pi A_{o_L,\pi} \otimes_{A_{o_L,\pi}} \mathcal{M} \to \pi\mathcal{M}$ is an isomorphism. Therefore, by Bourbaki's Flatness Criterion [Bou67, § III.5.2, Théorème 1(iii)], we may conclude that $\mathcal{M}$ is indeed flat over $A_{o_L,\pi}$.

6. Our next aim is to show that the kernel V of $\overline{F}\colon \sigma^*(\mathcal{M}/\pi\mathcal{M}) \to \mathcal{M}/\pi\mathcal{M}$ is trivial. We have already shown that $\mathfrak{J}^d\mathcal{M} \subseteq \mathrm{im}(F_\mathcal{M})$. Since $(z - \zeta) \in \mathfrak{J}$ for $\zeta := c^*(z) \in o_L$ we have a chain of $o_L\langle z\rangle$-modules $(z-\zeta)^d\mathcal{M} \subseteq \mathrm{im}(F_\mathcal{M}) \subseteq \mathcal{M}$. The element $\zeta \in o_L$ is zero mod π, and we obtain

$$z^d(\mathcal{M}/\pi\mathcal{M}) \subseteq \mathrm{im}(\overline{F}) \subseteq \mathcal{M}/\pi\mathcal{M}. \tag{5.2}$$

We know that $\mathcal{M}/\pi\mathcal{M}$ is finite free over $\ell[z]$. Therefore the middle term $W := \mathrm{im}(\overline{F})$ in the latter chain has full rank inside $\mathcal{M}/\pi\mathcal{M}$. Finally, taking ranks in the (split) short exact sequence of finite free $\ell[z]$-modules

$$0 \to V \to \sigma^*(\mathcal{M}/\pi\mathcal{M}) \xrightarrow{\overline{F}} W \to 0$$

accomplishes the proof that V indeed is trivial.

7. It remains to prove that the cokernel C of $F_{\mathcal{M}}\colon \sigma^*\mathcal{M} \to \mathcal{M}$ is a finite free o_L-module. In a first step we show that C is finitely generated over o_L. In equation (5.2) we saw that $C/\pi C$ is a quotient of $(\mathcal{M}/\pi\mathcal{M})\big/z^d(\mathcal{M}/\pi\mathcal{M})$ and hence a finite dimensional ℓ-vector space. Since $\pi \in \mathfrak{j}(A_{o_L,\pi})$ and since C, being a quotient of $\mathcal{M}$, is finitely generated over $A_{o_L,\pi}$, we conclude by Krull's Theorem [Mat86, Theorem 8.10] that C is π-adically separated. By [Mat86, Theorem 8.4] it follows that C is finitely generated.

In a second step we show that C is a flat o_L-module, which will imply that C is finite free over the local ring o_L. Since we have just seen that $C/\pi C$ is free and hence flat over ℓ, we only need to prove that C has trivial π-torsion. Then Bourbaki's Flatness Criterion [Bou67, § III.5.2, Théorème 1(iii)], will yield the desired result. So let $x \in \mathcal{M}$ with $\pi x = F_{\mathcal{M}}(y) \in \operatorname{im}(F_{\mathcal{M}})$ for an element $y \in \sigma^*\mathcal{M}$. Denoting residues modulo $\pi\mathcal{M}$ by a bar, we see that $0 = \overline{\pi x} = \overline{F}(\bar{y})$. By the injectivity of $\overline{F}$ we must have $y = \pi y'$ for a $y' \in \sigma^*\mathcal{M}$ and $x = F_{\mathcal{M}}(y') \in \operatorname{im}(F_{\mathcal{M}})$. Thus C is finite free over o_L and we have shown that $\underline{\mathcal{M}}$ is a good model for $\underline{M}_L$.

Conversely, in order to show that (i) implies (ii), suppose that $\mathcal{M}$ is a good model of M_L. We define

$$\widehat{\mathcal{M}} = \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)},$$

i.e. $\widehat{\mathcal{M}}$ equals the completion of $\mathcal{M}$ for the $(\varepsilon,\pi)A_{o_L,\pi}$-adic topology. It is clear that every fixed F-equivariant isomorphism of $A_{o_L,\pi}[1/\pi]$-modules $M_L \cong \mathcal{M}[1/\pi]$ gives rise to a natural F-equivariant $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-linear isomorphism $M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \widehat{\mathcal{M}}[1/\pi]$.

We claim that $\widehat{\mathcal{M}}$ is a local shtuka. Indeed, by base change, $\widehat{\mathcal{M}}$ is again locally free of finite rank. Furthermore, since the completion map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is Frobenius-equivariant and flat, we obtain an injective map $\widehat{\mathcal{M}} \otimes_{(A_{o_L,(\varepsilon,\pi)}),\sigma} A_{o_L,(\varepsilon,\pi)} \to \widehat{\mathcal{M}}$. Let C' be its cokernel, and let $C = \operatorname{coker}(F_{\mathcal{M}})$, i.e. $C' \cong C \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$. Since C is annihilated by $\mathfrak{J}^d$ the module C' equals C and it is finite free over o_L. Thus $\widehat{\mathcal{M}}$ is an effective local shtuka over o_L. $\square$

Corollary 5.4. *Let $\underline{M}_L$ be an analytic Anderson $A(1)$-motive over L. Then there is a (1:1)-correspondence*

$$\left\{\begin{array}{l}\textit{isomorphism classes of good}\\ \textit{models } \underline{\mathcal{M}} \textit{ of } \underline{M}_L \textit{ in the}\\ \textit{strong sense of Definition 4.6}\end{array}\right\} \overset{(1:1)}{\longleftrightarrow} \left\{\begin{array}{l}\textit{isomorphism classes of pairs } (\underline{\hat{M}}, f) \textit{ consisting of}\\ \bullet \textit{ a local shtuka } \underline{\hat{M}} \textit{ at } \varepsilon \textit{ over } o_L, \textit{ and}\\ \bullet \textit{ an isomorphism in } \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])\\ \quad f\colon \underline{M}_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi]\end{array}\right\}$$

$\underline{\mathcal{M}} \mapsto \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$, where on the right-hand side an isomorphism of pairs $(\underline{\hat{M}}, f) \xrightarrow{\sim} (\underline{\hat{N}}, g)$ is defined to be an isomorphism of local shtukas $\underline{\hat{M}} \to \underline{\hat{N}}$ which in the obvious manner is compatible with f and g.

Proof. Suppose that $\underline{\mathcal{M}}$ is a good model of $\underline{M}_L$. In the proof of 5.3 we have seen that its completion $\underline{\hat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ is a local shtuka at ε. Let $\mathcal{M}[1/\pi] \cong M_L$ be an F-equivariant isomorphism of $A_{o_L,\pi}[1/\pi]$-modules. It induces a natural isomorphism

$$f\colon (\mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi]) \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \widehat{\mathcal{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$$

which is F-equivariant, and satisfies $\mathcal{M} = \mathcal{M}[1/\pi] \cap f^{-1}(\widehat{\mathcal{M}})$, because $A_{o_L,\pi} = A_{o_L,\pi}[1/\pi] \cap A_{o_L,(\varepsilon,\pi)}$.

Conversely let a local shtuka $\underline{\hat{M}}$ together with an isomorphism $f\colon \underline{M}_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}}[1/\pi]$ be given. It remains to show that the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion $\underline{\hat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ of the good model $\mathcal{M} = M_L \cap f^{-1}(\hat{M})$ gained in the above construction gives back $\underline{\hat{M}}$. By construction of $\underline{\mathcal{M}}$, the map f restricts to an embedding $\mathcal{M} \hookrightarrow \hat{M}$, which in turn induces an F-equivariant and $A_{o_L,(\varepsilon,\pi)}$-linear map

$\psi\colon \hat{\mathcal{M}} \to \hat{M}$, which becomes an isomorphism after inverting π. Our aim is to show that already the map ψ is an isomorphism. We know that $\mathcal{M}$ is finite free over $o_L\langle z\rangle$ and that $\mathrm{rk}_{o_L[\![z]\!]}(\hat{\mathcal{M}}) = \mathrm{rk}_{o_L[\![z]\!]}(\hat{M}) =: s$. We fix an $o_L[\![z]\!]$-basis $\mathfrak{B}$ (resp., $\mathfrak{C}$) of $\hat{\mathcal{M}}$ (resp., of $\hat{M}$) and let $\mathbf{A} = {}_{\mathfrak{C}}[\psi]_{\mathfrak{B}} \in o_L[\![z]\!]^{s\times s}$ be the matrix which describes ψ with respect to $\mathfrak{B}$ and $\mathfrak{C}$. Likewise, we let

$$\mathbf{T} = {}_{\mathfrak{B}}[F_{\hat{\mathcal{M}}}]_{\sigma^*\mathfrak{B}}, \qquad \mathbf{T}' = {}_{\mathfrak{C}}[F_{\hat{M}}]_{\sigma^*\mathfrak{C}}$$

be the matrices corresponding to $F_{\hat{\mathcal{M}}}$ and $F_{\hat{M}}$, so that $\mathbf{AT} = \mathbf{T}'\sigma(\mathbf{A})$ by virtue of the F-equivariance of ψ. In order to see that ψ is an isomorphism, we need to show that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. To begin with, an elementary application of the Weierstraß Division Theorem for $o_L[\![z]\!]$ ([Bou67, VII.3.8.5]) shows that the kernel of the epimorphism $o_L[\![z]\!] \to o_L$, $z \mapsto \zeta$, is generated by $z - \zeta$, so that the latter is a prime element of $o_L[\![z]\!]$. Furthermore, recall that $o_L[\![z]\!]$, being a regular local ring, is factorial ([Mat86], 20.3). We know that $\hat{\mathcal{M}}$ is a local shtuka, so that $F_{\hat{\mathcal{M}}}$ becomes an isomorphism after inverting $z - \zeta$ which means that $\det(\mathbf{T})^{-1}$ lies in $o_L[\![z]\!][\frac{1}{z-\zeta}]$. Say we have a relation $(z-\zeta)^e = \det(\mathbf{T})u$ in $o_L[\![z]\!]$, for some $e \geq 0$ and some $u \in o_L[\![z]\!]$. By a comparison of powers of $z - \zeta$, we may assume that u is not divisible by $z - \zeta$. In this equation there is only one prime element of $o_L[\![z]\!]$ occuring on both sides, which, by factoriality, implies that u has to be a unit in $o_L[\![z]\!]$. Let $(z-\zeta)^{e'} = \det(\mathbf{T}')u'$ be the corresponding relation for the local shtuka $\hat{M}$, with a unit $u' \in o_L[\![z]\!]^\times$ and some suitable $e' \geq 0$. Since $\hat{\mathcal{M}} \to \hat{M}$ becomes an isomorphism after inverting π, we see that $\det(\mathbf{A}) \in o_L[\![z]\!][1/\pi]^\times$. Note that the natural reduction-mod-z map $o_L[\![z]\!] \to o_L$, $h \mapsto h(0)$, induces an epimorphism of abelian groups $o_L[\![z]\!][\frac{1}{\pi}]^\times \to L^\times$, so that the absolute term $\alpha := \det(\mathbf{A})(0)$ of $\det(\mathbf{A})$ lies in $L^\times$. By virtue of the relations derived above, the equation $\det(\mathbf{A})\det(\mathbf{T}) = \det(\mathbf{T}')\sigma(\det(\mathbf{A}))$ yields

$$\det(\mathbf{A})u^{-1}(z-\zeta)^e = u'^{-1}(z-\zeta)^{e'}\sigma(\det(\mathbf{A}))$$

which modulo z gives $\alpha^{q-1} = \frac{u'(0)}{u(0)}(-\zeta)^{e-e'}$ in $L^\times$. Suppose for a moment that $e = e'$ In this case it follows at once that α is a unit in o_L, so that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. Therefore it remains to verify that our assumption $e = e'$ is justified. This can be seen as follows: The reduction-mod-π map $o_L[\![z]\!] \to \ell[\![z]\!]$ is an epimorphism with kernel $\pi o_L[\![z]\!]$, and via applying the functor $\cdot \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$ to $F_{\hat{M}}\colon \sigma^*\hat{M} \to \hat{M}$ we obtain a commutative diagram

$$\begin{tikzcd}
\sigma^*\hat{M} = \hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!] \arrow[r] \arrow[d] & \hat{M} \arrow[d] \\
\bar{\sigma}^*\hat{M}/\pi\hat{M} = \hat{M}/\pi\hat{M} \otimes_{\ell[\![z]\!],\bar{\sigma}} \ell[\![z]\!] \arrow[r] & \hat{M}/\pi\hat{M}
\end{tikzcd}$$

where in the upper row (resp., the bottom row) both modules are finite free of the same rank over $o_L[\![z]\!]$ (resp., over $\ell[\![z]\!]$) and the arrow is given by $F_{\hat{M}}$ (resp., by $\bar{F} = F_{\hat{M}} \otimes \mathrm{id}_{\ell[\![z]\!]}$). The reduced matrix $\overline{\mathbf{T}'} \in \ell[\![z]\!]^{s\times s}$ describes the map $\bar{F}$ with respect to the $\ell[\![z]\!]$-bases $\overline{\sigma^*\mathfrak{C}} = \bar{\sigma}^*\bar{\mathfrak{C}}$ of $\bar{\sigma}^*\hat{M}/\pi\hat{M}$ and $\bar{\mathfrak{C}}$ of $\hat{M}/\pi\hat{M}$ respectively, and from what we have seen before, we derive the relation $\det(\overline{\mathbf{T}'})\overline{u'} = z^{e'}$, i.e. $e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'}))$, the latter being true since $\overline{u'} \in \ell[\![z]\!]^\times$. In particular we have $\det(\overline{\mathbf{T}'}) \in \ell[\![z]\!] - \{0\}$. A similar observation for the local shtuka $\hat{\mathcal{M}}$ instead of $\hat{M}$ shows that $e = \mathrm{ord}_z(\det(\overline{\mathbf{T}}))$. Let $C = \mathrm{coker}(F_{\hat{\mathcal{M}}})$ and $C' = \mathrm{coker}(F_{\hat{M}})$. Multiplication with the matrix $\overline{\mathbf{T}'}$ gives rise to a finite presentation $\ell[\![z]\!]^s \to \ell[\![z]\!]^s \to C'/\pi C' \to 0$. Taking determinants in an equation of the form $\mathbf{S}_1\overline{\mathbf{T}'}\mathbf{S}_2 = \mathrm{Diag}(a_1, ..., a_d, 0, 0, ..., 0)$, where $\mathbf{S}_1, \mathbf{S}_2 \in \mathrm{Gl}_s(\ell[\![z]\!])$ are suitable matrices such that $a_1, ..., a_d \in \ell[\![z]\!] - \{0\}$ are the elementary divisors of $\overline{\mathbf{T}'}$ (see [Bou81], VII.4.5.1), yields that necessarily $d = s$, so that $C'/\pi C'$ is a torsion $\ell[\![z]\!]$-module and

$$C'/\pi C' \cong \ell[\![z]\!]/a_1\ell[\![z]\!] \oplus ... \oplus \ell[\![z]\!]/a_s\ell[\![z]\!] \cong \ell^{n_1} \oplus ... \oplus \ell^{n_s}$$

where $n_j = \mathrm{ord}_z(a_j)$ and $\sum_j n_j = e'$, i.e. $e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'})) = \mathrm{rk}_\ell(C'/\pi C') = \mathrm{rk}_{o_L}(C')$, the latter equation being valid since $C'/\pi C' \cong C' \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$. Finally, imitating this argument for the local shtuka $\hat{\mathcal{M}}$ yields

that $e = \operatorname{ord}_z(\det(\overline{\mathbf{T}})) = \operatorname{rk}_\ell(C/\pi C) = \operatorname{rk}_{o_L}(C)$. So it remains to show that $\operatorname{rk}_{o_L}(C) = \operatorname{rk}_{o_L}(C')$. Indeed, we know that $\psi\colon \hat{\mathcal{M}} \to \hat{M}$ gives back f in the generic fiber, which means that ψ is an isomorphism after inverting π. Therefore, inverting π in the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \sigma^*(\hat{\mathcal{M}}) \arrow[r] \arrow[d, "\sigma^*\psi"'] & \hat{\mathcal{M}} \arrow[r] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \sigma^*\hat{M} \arrow[r] & \hat{M} \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

exhibits $(\sigma^*\psi)[1/\pi] = \sigma^*(\psi[1/\pi])$ and $\psi[1/\pi]$ as $o_L[\![z]\!][1/\pi]$-linear isomorphisms, so that the Snake Lemma yields $C'[1/\pi] \cong C[1/\pi]$, and we obtain $\operatorname{rk}_{o_L}(C') = \dim_L(C'[1/\pi]) = \dim_L(C[1/\pi]) = \operatorname{rk}_{o_L}(C)$, as desired. $\square$

6 The reduction criterion for Anderson motives

Definition 6.1. (a) Let $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$. Following Gardeyn [Gar03], $\underline{\mathcal{M}}$ is called A_{o_L}-*maximal* if for every $\underline{\mathcal{N}} \in \mathrm{FMod}(A_{o_L})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L})}(\underline{\mathcal{N}}, \underline{\mathcal{M}}) \to \mathrm{Hom}_{\mathrm{FMod}(A_L)}(\underline{\mathcal{N}}[1/\pi], \underline{\mathcal{M}}[1/\pi])$$

is surjective (and hence bijective).
(b) An object $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called $A_{o_L,\pi}$-*maximal* if for every $\underline{\mathcal{N}}' \in \mathrm{FMod}(A_{o_L,\pi})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi})}(\underline{\mathcal{N}}', \underline{\mathcal{M}}') \to \mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi}[1/\pi])}(\underline{\mathcal{N}}'[1/\pi], \underline{\mathcal{M}}'[1/\pi])$$

is surjective (and hence bijective).
(c) Let $\underline{M} \in \mathrm{FMod}(A_L)$. An object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ is called an A_{o_L}-*maximal model* for $\underline{M}$ if $\underline{\mathcal{M}}[1/\pi] \cong \underline{M}$ inside $\mathrm{FMod}(A_L)$ (i.e. $\underline{\mathcal{M}}$ is a *model* for $\underline{M}$) and if $\underline{\mathcal{M}}$ is A_{o_L}-maximal. Correspondingly, given $\underline{M}' \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, an object $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called an $A_{o_L,\pi}$-*maximal model* for $\underline{M}'$ if $\underline{\mathcal{M}}'[1/\pi] \cong \underline{M}'$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ and if $\underline{\mathcal{M}}'$ is $A_{o_L,\pi}$-maximal.

The existence of (A_{o_L}- and $A_{o_L,\pi}$-)maximal models has been established in [Gar03].

Proposition 6.2 ([Gar03, Proposition 2.13])**.** *Let $\underline{M} \in \mathrm{FMod}(A_L)$. Then the following assertions hold:*

(i) $\underline{M}$ admits an A_{o_L}-maximal model, which is unique up to unique isomorphism.

(ii) If a model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ of $\underline{M}$ is good in the weak sense of Definition 4.5, then it is A_{o_L}-maximal.

The next proposition is a variant of Gardeyn's theory of maximal models.

Proposition 6.3. *The following assertions hold:*

(i) Every $\underline{M} \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ admits a maximal model, which is unique up to unique isomorphism.

(ii) If $\underline{M} \in \mathrm{FMod}(A_L)$ is given and if $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ is an A_{o_L}-maximal model of $\underline{M}$ then $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \in \mathrm{FMod}(A_{o_L,\pi})$ is an $A_{o_L,\pi}$-maximal model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

(iii) Let $\underline{M} \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ and let $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ be a model of $\underline{M}$; if $\underline{\mathcal{M}}$ is a good model in the weak sense of Definition 4.5, then it is $A_{o_L,\pi}$-maximal.

Proof. For (i) (resp. (ii); resp. (iii)), see [Gar03], 3.3(i) (resp. 3.4(i); resp. 2.13(ii)). Note that strictly speaking Gardeyn proves these statements for the rings $\Gamma(\mathfrak{A}(\infty), \mathcal{O}_{\mathfrak{A}(\infty)})$ instead of $A_{o_L,\pi}[1/\pi]$ and $\Gamma(\mathfrak{A}(\infty), \mathcal{O}_{\mathfrak{A}(\infty)}) \cap A_{o_L,\pi}$ instead of $A_{o_L,\pi}$. His arguments carry over litteraly to our rings. $\square$

We may conclude:

Proposition 6.4. *A Frobenius A_L-module $\underline{M}$ admits a good model over A_{o_L} if and only if $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in$ $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ admits a good model over $A_{o_L,\pi}$. Then, up to isomorphism inside $\mathrm{FMod}(A_{o_L,\pi})$, a good model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ is given by $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ where $\underline{\mathcal{M}}$ is a good model of $\underline{M}$.*

Proof. First suppose that $\underline{M}$ admits a good model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$. It follows that $\underline{\mathcal{M}}$ is an A_{o_L}-maximal model of $\underline{M}$. Furthermore, its image $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ inside $\mathrm{FMod}(A_{o_L,\pi})$ is an $A_{o_L,\pi}$-maximal model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$. Since the reduction of $\underline{\mathcal{M}}$ is canonically isomorphic to the reduction of $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ by Proposition 4.8, it follows that the latter is a good model. Conversely, suppose that $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$. Necessarily $\underline{\mathcal{M}}'$ is a maximal model by Proposition 6.3(iii). We know that there is an A_{o_L}-maximal model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ of $\underline{M}$ such that $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \cong \underline{\mathcal{M}}'$, and that the reduction of $\underline{\mathcal{M}}'$ is canonically isomorphic to the reduction of $\underline{\mathcal{M}}$ by Propositions 6.2, 6.3(ii) and 4.8. Since $\underline{\mathcal{M}}'$ is a good model, so is $\underline{\mathcal{M}}$, which completes the proof. $\square$

If $\underline{M}$ is an Anderson A-motive we are more interested in good models in the strong sense of Definition 4.6. Then Proposition 6.4 has the following strong variant.

Theorem 6.5. *Let $\underline{M}$ be an Anderson A-motive over L. Then in the strong sense of Definition 4.6, $\underline{M}$ admits a good model $\underline{\mathcal{M}}$ if and only if the associated analytic Anderson $A(1)$-motive $\hat{\underline{M}} := \underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\underline{\mathcal{M}}'$. If this is the case then $\underline{\mathcal{M}}' \cong \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$.*

Proof. To prove one direction let $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ be a good model of $\underline{M}$ in the sense of Definition 4.6. We claim that the π-adic completion $\widehat{\underline{\mathcal{M}}} = \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ of $\underline{\mathcal{M}}$ is a good model for the analytic Anderson $A(1)$-motive $\underline{\mathcal{M}} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$. Since $A_{o_L,\pi}$ is flat over A_{o_L} we see that $\widehat{\underline{\mathcal{M}}}$ is locally free of finite rank over $A_{o_L,\pi}$, that $F_{\widehat{\mathcal{M}}} := F_{\mathcal{M}} \otimes \mathrm{id}$ is again injective and that $\mathrm{coker}(F_{\widehat{\mathcal{M}}}) = \mathrm{coker}(F_{\mathcal{M}})$ is finite free over o_L and annihilated by a power of $\mathfrak{J}$. So $\widehat{\underline{\mathcal{M}}}$ is a good model in the strong sense by Proposition 4.8.

Conversely, suppose that the analytification $\hat{\underline{M}}$ admits a good model $\underline{\mathcal{M}}'$ in the strong sense of Definition 4.6. In particular, by 6.4, the F-module $\underline{M}$ over A_L admits a good model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ in the weak sense of F-modules, Definition 4.5, and it remains to show that $\underline{\mathcal{M}}$ is a good model of $\underline{M}$ in the strong sense, i.e. that $C = \mathrm{coker}(F_{\mathcal{M}})$ is a finite free o_L-module and is annihilated by a power of the ideal $\mathfrak{J} \subseteq A_{o_L}$. We start with the latter claim. Let $\mathfrak{J}^d\mathrm{coker}(F_M) = 0$ and $\mathfrak{J}^d\mathrm{coker}(F_{\mathcal{M}'}) = 0$ for some integer d, and let $x \in \mathfrak{J}^d\mathcal{M}$. We need to show that $x \in \mathrm{im}(F_{\mathcal{M}})$. Since the good model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ as an F-module is uniquely determined up to unique isomorphism, we may by Proposition 6.4 assume that $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ (which is necessarily isomorphic to $\underline{\mathcal{M}}'$) is a good model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ in the strong sense. We remark that $\mathcal{M}$ is finite free over $o_L[z]$, say with finite basis $\mathfrak{B}$. Fixing an isomorphism $\underline{\mathcal{M}}[1/\pi] \cong \underline{M}$ inside $\mathrm{FMod}(A_L)$, the $o_L[z]$-basis $\mathfrak{B}$ of $\mathcal{M}$ induces an $L[z]$-basis on $\mathcal{M}[1/\pi]$ and hence on M, which in turn gives rise to a canonical induced basis on each remaining entry of the commutative diagram

$$\begin{tikzcd}
& \sigma^*(\mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle) \arrow[rr] \arrow[ddd, dashed] & & \mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle \arrow[ddd] \\
\sigma^*\mathcal{M} \arrow[ur] \arrow[rr] \arrow[d] & & \mathcal{M} \arrow[ur] \arrow[d] & \\
\sigma^*M \arrow[rr] \arrow[dr] & & M \arrow[dr] & \\
& \sigma^*(M \otimes_{L[z]} L\langle z\rangle) \arrow[rr] & & M \otimes_{L[z]} L\langle z\rangle
\end{tikzcd}$$

where each arrow is injective. Our chosen element $x \in \mathfrak{J}^d \mathcal{M}$ in particular lies in $\mathfrak{J}^d M$, so that there is a uniquely determined $y \in \sigma^* M$ such that $x = F_M(y)$. On the other hand, x gives rise to an element of $\mathcal{M}' \cong \mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle$. According to our assumption, we know that the cokernel of the map $F_{\mathcal{M}'}$ is annihilated by $\mathfrak{J}^d$. This implies that there is a uniquely determined element $y' \in \sigma^* \mathcal{M}'$ which is mapped to (the image of) x in $\mathcal{M}'$. Finally, since y' is necessarily mapped to (the image of) y via the dashed vertical arrow, writing y' in terms of the $o_L\langle z\rangle$-basis induced by $\mathfrak{B}$ and keeping track of linear combinations shows that the coefficients of y' have, in fact, to lie inside $o_L\langle z\rangle \cap L[z] = o_L[z]$, which proves that $\mathfrak{J}^d C = 0$. In particular, C is finitely generated over o_L.

It remains to see that C does not have π-torsion. In order to prove this, we need to see that $\pi x \in \mathrm{im}(F_{\mathcal{M}})$ for a given $x \in \mathcal{M}$ implies $x \in \mathrm{im}(F_{\mathcal{M}})$. We again use that $\mathcal{M}$ is finite free over $o_L[z]$ and remark that, since $\underline{\mathcal{M}}$ is a good model of $\underline{M}$ as an F-module, the bottom horizontal arrow in the commutative diagram

$$\begin{tikzcd} \sigma^*\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \bar{\sigma}^*(\mathcal{M} \otimes_{o_L[z]} \ell[z]) \arrow[r] & \mathcal{M} \otimes_{o_L[z]} \ell[z] \end{tikzcd}$$

is injective. Furthermore, the vertical maps are surjective and in the upper (resp. bottom) row both modules are finite free over $o_L[z]$ (resp. over $\ell[z]$) of the same rank. From $\pi x \in \mathrm{im}(F_{\mathcal{M}})$ it follows that there is a uniquely determined $y \in \sigma^*\mathcal{M}$ such that $\pi x = F_{\mathcal{M}}(y)$. Since πx goes to zero under the right-hand projection, necessarily y has to go to zero via the left-hand projection. A chosen $o_L[z]$-basis of $\mathcal{M}$ induces bases of each of the other entries of the above diagram. Keeping track of coefficients in linear combinations one verifies that $y \in \pi\sigma^*\mathcal{M}$. Finally, since $\mathcal{M}$ is torsion-free, we obtain $x = F_{\mathcal{M}}(y)$, as desired. So, for example, by [Bou67, § III.5.2, Théorème 1(iii)], we may conclude that C is flat over o_L. □

Theorem 5.3 implies the following criterion for good reduction of Anderson A-motives, which can be regarded as an analog of Grothendieck's [SGA 7, Proposition IX.5.13] and de Jong's [dJ98, 2.5] reduction criteria for abelian varieties.

Corollary 6.6. *Let $\underline{M}$ be an Anderson A-motive over L such that* $\mathrm{coker}(F_{\underline{M}})$ *is annihilated by $\mathfrak{J}^d$ say. Then the following assertions are equivalent:*

(i) $\underline{M}$ admits a good model $\underline{\mathcal{M}}$ in the strong sense of Definition 4.6, i.e. there is an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ such that $\mathrm{coker}(F_{\mathcal{M}})$ *is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$, together with an isomorphism $\underline{\mathcal{M}}[1/\pi] \cong \underline{M}$ inside* $\mathrm{FMod}(A_L)$*;*

(ii) There is an effective local shtuka $\underline{\hat{M}}$ at ε over o_L such that $\mathrm{coker}(F_{\hat{M}})$ *is annihilated by $\mathfrak{J}^d$, and an isomorphism $\underline{M} \otimes_{A_L} A_{o_L,\varepsilon}[1/\pi] \cong \underline{\hat{M}}[1/\pi]$ inside* $\mathrm{FMod}(A_{o_L,\varepsilon}[1/\pi])$.

In particular, we obtain a one-to-one correspondence between (isomorphism classes of) good models of $\underline{M}$ and (isomorphism classes of) pairs $(\underline{\hat{M}}, f)$ consisting of a local shtuka $\underline{\hat{M}}$ at ε over o_L and an isomorphism $f\colon \underline{M} \otimes_{A_L} A_{o_L,\varepsilon}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi]$ inside $\mathrm{FMod}(A_{o_L,\varepsilon}[1/\pi])$. □

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Urs Hartl
Universität Münster
Mathematisches Institut
Einsteinstr. 62
D – 48149 Münster
Germany
www.math.uni-muenster.de/u/urs.hartl/

Simon Hüsken
Universität Münster
Mathematisches Institut
Einsteinstr. 62
D – 48149 Münster
Germany